# Dynamic Semantics of $\tau$N-Theories

## Turing Centenary Conference

22 June 2012


Ralph L. Wojtowicz
Baker Mountain Research Corporation
Yellow Spring, WV USA

Shepherd University
Shepherdstown, WV USA

Selmer Bringsjord
Rensselaer Polytechnic Institute
Troy, NY USA

John Hummel
University of Illinois
Champaign-Urbana, IL USA

# An Example

1. The set $\mathbb{R}$ of real numbers is uncountable.

   For $x_i = 0.\,x_{i,1}\,x_{i,2}\cdots \in (0,1)$, define $y_j = (x_{j,j}+1) \bmod 10$.

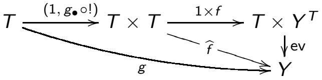
2. There is a partial function $\mathbb{N} \rightharpoonup \mathbb{N}$ that is not Turing computable.

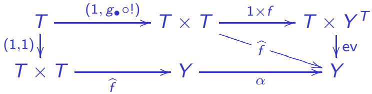
   $$g(n) = \begin{cases} f_n(n)+1 & \text{if } f_n(n) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

3. If $\mathbb{T}$ is a consistent, recursively presented, first-order theory containing a recursive interpretation of PA, then $\mathbb{T}$ is incomplete.

   $\exists x\,((x = \ulcorner\varphi\urcorner) \wedge \varphi)$ $\qquad \vdash_{\mathbb{T}} (G_\varphi \iff \varphi(\ulcorner G_\varphi \urcorner / y))$ $\qquad \varphi = \neg\chi_{\text{Gödel numbers of theorems}}$

4. Let $Y$ be an object of a cartesian closed category. If there is an object $T$ and a morphism $f : T \to Y^T$ such that for any $g : T \to Y$, there is a $g_\bullet : 1 \to T$ for which the diagram below commutes, then every $\alpha : Y \to Y$ has a fixed point.



# Analogical Reasoning

**Idea**: Design a computational system with the capacity to utilize the analogies we perceive among the four theorems on the previous slide.

- Analogical Reasoning: The (possibly) unique capacity of humans to formulate and reason with relations using indexing and substitution.
  - Hummel and Holyoak: Relational reasoning in a neurally-plausible cognitive architecture: An overview of the LISA project. 2003
- Transfer Learning: When learning tasks over its lifetime, a system employs knowledge gathered in previous, related tasks to improve performance.
  - Lifelong Robot Learning (S. Thrun and T. M. Mitchell) 1993
  - Survey: L. Torrey and J. Shavlik. *Handbook of Research on Machine Learning Applications*. 2009
  - Deep Transfer via $2^{nd}$-Order Markov Logic. (P. Domingos et al.) 2009

**Approach**: Explore concepts and techniques of categorical logic that can be used to model and implement analogical reasoning involving the diagonal lemma and Peano Arithmetic.

# $\tau$N-Theories — Syntax

- Signature
  - Types: sorts, 1, $A \times B$, $N$, $PA$
  - Function and relation symbols
- Terms
  - Variables $x : A$
  - Function application $f(t) : B$ if $f : A \to B$ and $t : A$
  - Products: $* : 1$, $\langle s, t \rangle : A \times B$ for $s : A$ and $t : B$ and
    $\mathrm{fst}(z) : A$ and $\mathrm{snd}(z) : B$ for $z : A \times B$
  - Natural number: $0 : N$, $\mathrm{succ}(t) : N$ if $t : N$ and $\mathrm{iter}_x(m, a, n) : A$ if $m : A$,
    $a : A$ and $n : N$ with $x$ not free in $a$ or $n$ (or in $\mathrm{iter}_x(m, a, n)$)
  - Power: $\{x : A \mid \varphi\} : PA$ (with $\mathrm{FV}(\varphi)/\{x\}$ as set of free variables)
- Formulae
  - Atomic: $R(t)$, $(t =_A s)$ and $(s \in_A t)$ for $s : A$ and $t : PA$
  - Compound: $\varphi * \psi$ with $*$ one of $\wedge$, $\vee$, $\Rightarrow$
  - Negated: $\neg\varphi$
  - Quantified: $(\forall x)\varphi$ and $(\exists x)\varphi$

# $\tau$N-Theories — Sequent Calculus

**Structural Rules**[1]

$$(\varphi \vdash_{\vec{x}} \varphi) \qquad \frac{(\varphi \vdash_{\vec{x}} \psi)}{(\varphi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}])} \qquad \frac{(\varphi \vdash_{\vec{x}} \psi) \ (\psi \vdash_{\vec{x}} \chi)}{(\varphi \vdash_{\vec{x}} \chi)}$$

**Implication**

$$\frac{((\varphi \wedge \psi) \vdash_{\vec{x}} \chi)}{(\varphi \vdash_{\vec{x}} (\psi \Rightarrow \chi))}$$

**Equality**

$$(\top \vdash_{x} (x = x)) \qquad ((\vec{x} = \vec{y}) \wedge \varphi \vdash_{\vec{z}} \varphi[\vec{y}/\vec{x}])$$

**Quantification**[2]

$$\frac{(\varphi \vdash_{\vec{x},y} \psi)}{((\exists y)\varphi \vdash_{\vec{x}} \psi)} \qquad \frac{(\varphi \vdash_{\vec{x},y} \psi)}{(\varphi \vdash_{\vec{x}} (\forall y)\psi)}$$

**Conjunction**

$$(\varphi \vdash_{\vec{x}} \top) \qquad ((\varphi \wedge \psi) \vdash_{\vec{x}} \varphi) \qquad ((\varphi \wedge \psi) \vdash_{\vec{x}} \psi) \qquad \frac{(\varphi \vdash_{\vec{x}} \psi) \ (\varphi \vdash_{\vec{x}} \chi)}{(\varphi \vdash_{\vec{x}} (\psi \wedge \chi))}$$

**Disjunction**

$$(\bot \vdash_{\vec{x}} \varphi) \qquad (\varphi \vdash_{\vec{x}} (\varphi \vee \psi)) \qquad (\psi \vdash_{\vec{x}} (\varphi \vee \psi)) \qquad \frac{(\varphi \vdash_{\vec{x}} \chi) \ \ (\psi \vdash_{\vec{x}} \chi)}{((\varphi \vee \psi) \vdash_{\vec{x}} \chi)}$$

**Product**

$$(\top \vdash_{x} (x =_1 *)) \quad (\top \vdash_{x,y} (\mathrm{fst}(\langle x, y\rangle) = x)) \qquad (\top \vdash_{z} (\langle \mathrm{fst}(z), \mathrm{snd}(z)\rangle = z))$$

$$(\top \vdash_{x,y} (\mathrm{snd}(\langle x, y\rangle) = y))$$

**Power**[3]

$$(\top \vdash_{w} (w =_{PA} \{x : A \,|\, x \in_A w\})) \qquad ((z \in_A \{y : A \,|\, \varphi\}) \dashv\vdash_{\vec{x},z} \varphi[z/y])$$

**Natural Numbers**

$$(\top \vdash_{\vec{y}} (\mathrm{iter}_x(m, a, 0) = a)) \quad (\top \vdash_{\vec{y}} (\mathrm{iter}_x(m, a, \mathrm{succ}(n)) = m[\mathrm{iter}_x(m, a, n)/x]))$$

$$(((0 \in_N z) \wedge (\forall y)((y \in_N z) \Rightarrow (\mathrm{succ}(y) \in_N z))) \vdash_{z:PN} (\forall y)(y \in_N z))$$

Contexts are suitable for the formulae that occur on both sides of $\vdash$.

1 In the substitution rule, $\vec{y}$ contains all the variables of $\vec{x}$.

2 Bound variables do not also occur free in any sequent.

3 $w : PA$ is a variable.

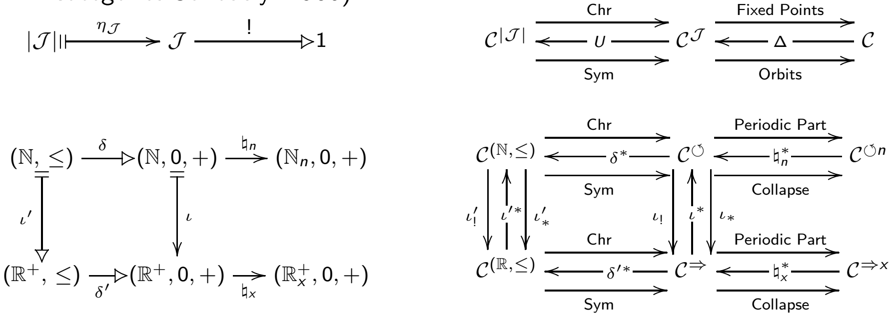
# $\tau$N-Theories — Models and Morphisms

- Any topos with natural number object is a suitable semantic category.
  - Soundness: If $\sigma$ is provable in $\mathbb{T}$, then it is satisfied in all $\mathbb{T}$-models in such toposes. D4.3.17
  - Completeness: If $\sigma$ is satisfied in all $\mathbb{T}$-models in such toposes, then it is provable. D4.3.19(b)
  - Peano Arithmetic: Any such topos has a model of PA. A2.5.4, A2.5.5
  - Recursive Partial Functions: $\mathbb{N}^k \rightharpoonup \mathbb{N}$ have interpretations.
- Logical Functors: cartesian and preserves exponentials, $\Omega$ and $N$
  - Preserve satisfaction of $\tau$N sequents
- Geometric morphisms: adjoint pairs $\begin{array}{c}\mathcal{F}\\ f^*\uparrow\downarrow f_*\\ \mathcal{E}\end{array}$ with $f^*$ cartesian
  - Preserve satisfaction of Horn sequents of $\mathcal{F}$ $(\top, \wedge)$
  - Preserve satisfaction of regular sequents of $\mathcal{E}$ $(\top, \wedge, \exists)$
  - Reflect natural number objects of $\mathcal{E}$

Citations in green are from Johnstone's *Sketches of an Elephant*. 2002.

# Dynamic Semantics

- Any presheaf topos $[\mathcal{C}, \mathrm{Set}]$ has a natural number object.
- Kan extensions are a source of (essential) geometric morphisms.
- Lawvere elucidated the dynamic systems examples shown below (Taking Categories Seriously. 1986).




- $U$, $\Delta$, $\iota^*$, $\iota'^*$, $\delta^*$, $\delta'^*$, $\natural^*_n$ and $\natural^*_x$ preserve satisfaction of coherent sequents ($\top$, $\wedge$, $\exists$, $\bot$, $\vee$)

# Kan Extension Calculations

$$\text{Discrete Iterators} \quad \mathbf{Sets}^{\circlearrowleft} \begin{array}{c} \xrightarrow{\iota_*} \\ \xleftarrow{\iota^*} \\ \xrightarrow{\iota_!} \end{array} \mathbf{Sets}^{\Rightarrow} \quad \text{Discrete Flows}$$

| | | |
|---|---|---|
| $\iota_*$ | $(X, f)$ | $\{h \in X^{\mathbb{R}^+} \mid h(r+1) = f(h(r))\}$ |
| $\iota^*$ | $(X, f(\cdot, 1))$ | $(X, f)$ |
| $\iota_!$ | $(X, f)$ | $X \times \mathbb{R}^+/\sim,\ (x, t+n) \sim (f^n(x), t)$ |
| $\Omega$ | $\top \circlearrowleft 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots \quad \infty \circlearrowleft$ | $\top \circlearrowleft 0 \longleftarrow \longleftarrow \longleftarrow \cdots \quad \infty \circlearrowleft$ |
| $N$ | $(\mathbb{N}, \mathrm{id})$ | $(\mathbb{N}, (n, t) \mapsto n)$ |
| $H_*$ | $0 \longrightarrow 1 \longrightarrow 2 \cdots\rightarrow$ | $0 \longrightarrow \longrightarrow \longrightarrow \cdots$ |
| $P\mathcal{X}$ | subobjects of $H_* \times \mathcal{X}$ | subobjects of $H_* \times \mathcal{X}$ |

- None of $\iota_*$, $\iota^*$ or $\iota_!$ is logical:
  - $\iota_*$ has a left but no right adjoint (it does not preserve coproducts).
  - $\iota^*$ does not preserve $\Omega$.
  - $\iota_!$ has a right but no left adjoint (it does not preserve 1).

# Prospective Tools

- Under what conditions is a geometric morphism induced by Kan extensions a logical functor?
- Explore sheaf examples
- Source of logical functors: $\mathcal{E}/B \xrightarrow{f^*} \mathcal{E}/A$ for $A \xrightarrow{f} B$ in a topos $\mathcal{E}$
- Suitability of the effective topos
- Morita morphism: relate $\tau$N-theories that have may distinct signatures

$$\begin{array}{ccccc}
\mathbb{T} & \quad & \mathbb{T}\text{-Mod}(\mathcal{E}) \xleftarrow{\approx} \square(\mathcal{C}_{\mathbb{T}}, \mathcal{E}) & \quad & \mathcal{C}_{\mathbb{T}} \\
\downarrow & & \uparrow \qquad\qquad\qquad \uparrow & & \downarrow \\
\mathbb{T}' & & \mathbb{T}'\text{-Mod}(\mathcal{E}) \xleftarrow{\approx} \square(\mathcal{C}_{\mathbb{T}'}, \mathcal{E}) & & \mathcal{C}_{\mathbb{T}'}
\end{array}$$

where $\square =$ Log or some other class of functors.